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Complex phase space formulation of supersymmetric quantum mechanics: analysis of shape-invariant potentials

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Abstract
With a view to increasing the scope of applications of supersymmetric quantum mechanics (SUSY QM), we formulate the same in a complex phase space. Within this framework, the concept of shape invariance is reinvestigated and an insight into the eigenvalue spectra of non-Hermitian Hamiltonians is sought. The results are applied to a variety of potentials. We claim that the shape invariance for these potentials in the complex phase space can be retrieved in terms of the prescriptions already proposed in the conventional SUSY QM, in that the transformation of potential parameters takes the form of a reflection in the parameter space. Interestingly, some of these features turn out to be the generalization of the concept of quasi-parity used recently in the context of SUSY QM of PT-symmetric potentials. Further, a correspondence of the present approach with other complex formulations of SUSY QM and also in two real dimensions is demonstrated.

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1. Introduction

During the last two decades or so, supersymmetric (SUSY) quantum mechanics (QM) has evolved [1, 2] as an alternative tool to investigate the potential problems of Schrodinger quantum mechanics. As such SUSY QM is much richer in mathematical content than it is in providing physical insight into a particular problem. Nonetheless, SUSY QM has suggested several interesting features with regard to the eigenvalue spectra and eigenfunctions for a variety of potentials. For example, the property of shape invariance of a potential manifests much better in this approach. Also, the solution of the second-order Schrodinger equation for a given potential now reduces to the handling of the first-order Riccati equation involving the 'superpotential', $W(x)$, which is linked directly with the Riccati transformation. In addition to this, several other extensions and applications of the methods of SUSY QM have generated considerable interest in recent years [3–10].

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Recently, there have been concerted efforts to study non-Hermitian SUSY QM in the context of PT-symmetric [10–13] and pseudo-Hermitian [23, 24] Hamiltonians. In fact, the PT-symmetric version based on the space–time reflection symmetry of a complex Hamiltonian is found [9, 11–22] to admit real eigenvalues for certain parametric domains. This fact has given a great impetus in the recent past to study the QM of non-Hermitian Hamiltonians which otherwise has been of minimal interest from the historical perspective in the study of physical problems. The idea initiated by Bender and co-workers [14–16] has now gained considerable currency as evidenced by the subsequent work of several other authors [17–21].

While the reality of eigenvalues for a variety of PT-invariant Hamiltonians has been the main focus of the study, efforts have also been made to investigate other general aspects [23–26] of the related non-Hermitian PT-symmetric QM. For example, within this framework, the problem pertaining to the normalization of the eigenfunctions [23–25] in terms of the equation of continuity [24], the concept of indefinite metric [24, 25] and the violation of unitarity [26] are investigated. Mostafazadeh [27] has suggested another version of the non-Hermitian Hamiltonian H (termed pseudo-Hermitian) that also gives rise to real eigenvalues. Compared to the PT-symmetric Hamiltonian, in this case however there is somewhat less departure from the non-Hermiticity of H. In fact, H is defined to be pseudo-Hermitian with respect to an operator η if $H^{+} = \eta H \eta^{-1}$, where η is a linear, Hermitian and invertible operator. Very recently, supersymmetric and several other aspects of pseudo-Hermitian QM have also been studied [28, 29].

With regard to the study of non-Hermitian Hamiltonian systems, we have been pursuing a complex phase space approach to study the classical [30, 31] and quantum [32, 33] aspects of certain systems. In this approach, besides considering a complex phase space characterized by [30–34]

$$x = x_1 + \mathrm{i}p_2 \qquad p = p_1 + \mathrm{i}x_2 \tag{1}$$

the complexity arising out of the underlying parameters of the system has also been incorporated [33] as an essential ingredient. As a result the reality of eigenvalue spectra in the Schrodinger approach to QM is demonstrated to arise in ways more than one and the quasi-exact solutions of an analogous Schrodinger equation are investigated for a variety of potentials. In the classical context, on the other hand, the methods for the construction of complex invariants for such systems are investigated [30] and the integrability of several analogous two-dimensional real Hamiltonian systems has been addressed [31] within this framework. In the present work, we demonstrate the viability of this approach to study the non-Hermitian SUSY QM.

In the present approach the problem is more involved since we are dealing with a non-Hermitian operator of a very general nature in addition to a complex parameter space (cf [33]). In fact, the Hamiltonian $H(x, p)$ is now a function of two complex variables (cf equation (1)). It is a different matter that p in the quantum domain is expressed as $(-\mathrm{i}\,\mathrm{d}/\mathrm{d}x)$ and the use of Cauchy–Rienmann conditions on the derivatives of the eigenfunctions leads to considerable simplifications (cf [32, 33]). It is true that the definition of the eigenvalue equation $H\psi(x) = E\psi(x)$ in our approach needs further elucidation in terms of the space of non-Hermitian operators, but interestingly several results, derived for the sextic and other potentials (cf [33]) in the present setup, reduce as a special case to those obtained for the corresponding PT-symmetric systems. No doubt, in our approach, the eigenvalues in general are complex but there is enough scope to make them real in terms of constraining relations satisfied by the complex parameters (for details see [33]).

In the study of the non-Hermitian SUSY QM of the PT-symmetric spiked harmonic oscillator and Scarf II potentials, Znojil [35] and Levai and Znojil [36] have noticed the duality of normalizable states in terms of what is known as the concept of quasi-parity. These features of the PT-symmetry-inspired SUSY QM will, however, appear conspicuously and unambiguously in a very general manner (cf section 6) in the present approach. The arrangement of the paper is as follows. We recapitulate in section 2 the results of the conventional SUSY QM for the purpose of ready comparison, before formulating the SUSY QM in the complex phase space in section 3. In section 4, we discuss briefly the complex version of the shape-invariance property of the potentials and then investigate in section 5 the ground-state features for a variety of superpotentials. In section 6, we attempt to find some possible connections between the results obtained in the present approach and those obtained using other formulations of SUSY QM, particularly in higher dimensions. Despite the fact that our approach is quite general and the results obtained here in some cases reduce to those obtained using other (PT-symmetric or pseudo-Hermitian) approaches as special limiting cases, several intriguing problems still remain to be addressed. Some of these difficulties, such as the question concerning the orthonormality of eigenstates, are discussed in section 7. Finally, concluding remarks are made in section 8.

2. Salient features of conventional SUSY QM

In SUSY QM, the superpotential $W(x)$ derived from the ground-state wavefunction, ψ_0, for the potential $V^{(-)}(x)$ having zero ground-state energy, is defined as

$$W(x) = -\psi_0'/\psi_0 \tag{2a}$$

or

$$\psi_0(x) = N_0 \exp\left(-\int^x W(y)\,\mathrm{d}y\right). \tag{2b}$$

Knowledge of the ground-state energy E_0 and superpotential $W(x)$ allows us to factorize the Hamiltonian H in the following form [2]:

$$2H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) = A^+A + 2E_0 \tag{3a}$$

where

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W \qquad A^+ = -\frac{\mathrm{d}}{\mathrm{d}x} + W. \tag{3b}$$

The pair of Hamiltonians $H^{(\pm)}$ related by supersymmetry is given by

$$H^{(\pm)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(\pm)}$$

where the supersymmetric partner potentials $V^{(\pm)}$ are related via

$$V^{(\pm)} = W^2 \pm W' \tag{4}$$

with

$$H^{(-)}\psi_0 = 0. \tag{4$'$}$$

It should be noted that $W^2(x)$ is the average of potentials $V^{(+)}(x)$ and $V^{(-)}(x)$ whereas W' is proportional to the commutator of A and A^+. The eigenvalues and eigenvectors of the two Hamiltonians $H^{(-)}$ and $H^{(+)}$ are related by

$$\begin{aligned} &E_0^{(-)} = 0 \qquad E_{n+1}^{(-)} = E_n^{(+)} \\ &\psi_n^{(+)} = \left[E_{n+1}^{(-)}\right]^{-1/2} A\psi_{n+1}^{(-)} \\ &\psi_{n+1}^{(-)} = \left[E_n^{(+)}\right]^{-1/2} A^+\psi_n^{(+)} \end{aligned} \tag{5}$$

where $n = 0, 1, 2, \ldots$. The above relations between the eigenvalues and eigenfunctions of the two Hamiltonians ensure that all the eigenfunctions of $H^{(+)}$ can be obtained if the eigenfunctions of $H^{(-)}$ are known. Further, the superpartner potentials $V^{(+)}$ and $V^{(-)}$ have the same energy spectrum except that the ground-state energy $E_0^{(-)}(= 0)$ of $V^{(-)}$ has no corresponding level for $V^{(+)}$.

3. Formulation of complex SUSY QM

In this section, we attempt to reformulate the SUSY QM in the complex phase space so as to get deeper insight into the nature of eigenvalues and eigenfunctions of the resulting non-Hermitian Hamiltonians. Our endeavour is to look for appropriate connections between the eigenvalues and eigenfunctions of such non-Hermitian potentials. In other words, our prescriptions apply to any potential function of a complex variable.

3.1. Operator formulation of complex SUSY QM

Consider a complex Hamiltonian $H^{(-)}(x)$ with x given by (1) and the corresponding eigenfunction $\psi_0^{(-)}(x) = \psi_0(x)$ with the eigenvalue $E_0^{(-)} = 0$. In view of (1), we write $H^{(-)}$, $\psi_0(x)$ and $E_0^{(-)}$ in terms of their real and imaginary parts (labelled with subscripts r and i, respectively)

$$H^{(-)} = H_{\rm r}^{(-)} + {\rm i}H_{\rm i}^{(-)} \qquad \psi_0 = \psi_{0\rm r} + {\rm i}\psi_{0\rm i} \qquad E_0^{(-)} = E_{0\rm r}^{(-)} + {\rm i}E_{0\rm i}^{(-)}. \tag{6}$$

Thus the Schrödinger equation (4′) for the ground state in the complex phase space takes the form

$$\begin{aligned}&\big(H_{\rm r}^{(-)}\psi_{0\rm r} - H_{\rm i}^{(-)}\psi_{0\rm i}\big) + {\rm i}\big(H_{\rm i}^{(-)}\psi_{0\rm r} + H_{\rm r}^{(-)}\psi_{0\rm i}\big)\\ &\quad = -\left(\frac{\partial^2\psi_{0\rm r}}{\partial x_1^2} - \frac{\partial^2\psi_{0\rm r}}{\partial p_2^2} + \frac{2\partial^2\psi_{0\rm i}}{\partial x_1\partial p_2}\right) + V_{\rm r}^{(-)}\psi_{0\rm r} - V_{\rm i}^{(-)}\psi_{0\rm i}\\ &\qquad + {\rm i}\left\{-\left(\frac{\partial^2\psi_{0\rm i}}{\partial x_1^2} - \frac{\partial^2\psi_{0\rm i}}{\partial p_2^2} - \frac{2\partial^2\psi_{0\rm r}}{\partial x_1\partial p_2}\right) + V_{\rm r}^{(-)}\psi_{0\rm i} + V_{\rm i}^{(-)}\psi_{0\rm r}\right\} = 0\end{aligned}$$

which, after equating the real and imaginary parts in both the above equations separately to zero, yields a pair of equations, namely

$$\frac{\partial^2\psi_{0\rm r}}{\partial x_1^2} - \frac{\partial^2\psi_{0\rm r}}{\partial p_2^2} + \frac{2\partial^2\psi_{0\rm i}}{\partial x_1\partial p_2} + V_{\rm r}^{(-)}\psi_{0\rm r} - V_{\rm i}^{(-)}\psi_{0\rm i} = 0$$

$$\frac{\partial^2\psi_{0\rm i}}{\partial x_1^2} - \frac{\partial^2\psi_{0\rm i}}{\partial p_2^2} - \frac{2\partial^2\psi_{0\rm r}}{\partial x_1\partial p_2} + V_{\rm r}^{(-)}\psi_{0\rm i} + V_{\rm i}^{(-)}\psi_{0\rm r} = 0$$

along with the real and imaginary parts of the Hamiltonian $H^{(-)}$ as

$$H_{\rm r}^{(-)} = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial p_2^2} + V_{\rm r}^{(-)} \tag{7a}$$

$$H_{\rm i}^{(-)} = -\frac{2\partial^2}{\partial x_1\partial p_2} + V_{\rm i}^{(-)}. \tag{7b}$$

Alternatively, the Schrödinger equation (4′) for the ground state can also be written as

$$\frac{1}{\psi_0}\frac{{\rm d}^2\psi_0}{{\rm d}x^2} = V^{(-)}$$

which, in turn, leads to

$$V_{\rm r}^{(-)} + {\rm i}V_{\rm i}^{(-)} = \frac{1}{(\Psi_{0{\rm r}} + {\rm i}\Psi_{0{\rm i}})}\left[\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial p_2^2} - \frac{2{\rm i}\partial^2}{\partial x_1 \partial p_2}\right](\Psi_{0{\rm r}} - {\rm i}\Psi_{0{\rm i}}).$$

Now, as a physical requirement we make use of the analyticity property of $\psi_0(x)$ via the Cauchy–Riemann conditions

$$\frac{\partial \psi_{0{\rm r}}}{\partial x_1} = \frac{\partial \psi_{0{\rm i}}}{\partial p_2} \qquad \frac{\partial \psi_{0{\rm i}}}{\partial x_1} = -\frac{\partial \psi_{0{\rm r}}}{\partial p_2}. \tag{8}$$

This yields the expression for $V_{\rm r}^{(-)}$ and $V_{\rm i}^{(-)}$ as

$$V_{\rm r}^{(-)} = \frac{4}{|\psi_0|^2}(\psi_{0{\rm r}}''\psi_{0{\rm r}} + \psi_{0{\rm i}}''\psi_{0{\rm i}}) \qquad V_{\rm i}^{(-)} = \frac{4}{|\psi_0|^2}(\psi_{0{\rm i}}''\psi_{0{\rm r}} - \psi_{0{\rm r}}''\psi_{0{\rm i}}) \tag{9}$$

where

$$\psi_{0{\rm r}}' = \frac{\partial \psi_{0{\rm r}}}{\partial x_1} \qquad \text{and} \qquad |\psi_0|^2 = \psi_{0{\rm r}}^2 + \psi_{0{\rm i}}^2.$$

Further use of these results in equation (7) gives

$$H_{\rm r}^{(-)} = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial p_2^2} + \frac{4}{|\psi_0|^2}(\psi_{0{\rm r}}''\psi_{0{\rm r}} + \psi_{0{\rm i}}''\psi_{0{\rm i}}) \tag{10a}$$

$$H_{\rm i}^{(-)} = \frac{2\partial^2}{\partial x_1 \partial p_2} + \frac{4}{|\psi_0|^2}(\psi_{0{\rm i}}''\psi_{0{\rm r}} - \psi_{0{\rm r}}''\psi_{0{\rm i}}). \tag{10b}$$

Note that the terms in the parentheses of equations (10*a*) and (10*b*) can be written in terms of the modulus and argument of ψ_0 and their partial derivatives so as to read

$$\psi_{0{\rm r}}''\psi_{0{\rm r}} + \psi_{0{\rm i}}''\psi_{0{\rm i}} = \tfrac{1}{2}(|\psi_0|^2)'' - |\psi_0'|^2 \tag{11a}$$

$$\psi_{0{\rm r}}\psi_{0{\rm i}}'' - \psi_{0{\rm i}}\psi_{0{\rm r}}'' = |\psi_0|^2\left\{\left(\tan^{-1}\frac{\psi_{0{\rm i}}}{\psi_{0{\rm r}}}\right)'' + \frac{1}{|\psi_0|^2}(|\psi_0|^2)'\left(\tan^{-1}\frac{\psi_{0{\rm i}}}{\psi_{0{\rm r}}}\right)'\right\} \tag{11b}$$

where the primes, as usual, denote the partial derivatives w.r.t. x_1 and $|\psi_0'|^2 = \psi_{0{\rm r}}'^2 + \psi_{0{\rm i}}'^2$. If we denote the argument of eigenfunction ψ_0 by α, namely $\alpha = \tan^{-1}\frac{\psi_{0{\rm i}}}{\psi_{0{\rm r}}}$, then equations (10*a*) and (10*b*) can be recast as

$$H_{\rm r}^{(-)} = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial p_2^2} + \frac{2}{|\psi_0|^2}(|\psi_0|^2)'' - \frac{4|\psi_0'|^2}{|\psi_0|^2} \tag{12a}$$

$$H_{\rm i}^{(-)} = \frac{2\partial^2}{\partial x_1 \partial p_2} + 4\alpha'' + \frac{4(|\psi_0|^2)'}{|\psi_0|^2}\alpha'. \tag{12b}$$

We now define the operators

$$A_{\rm r} = \frac{\partial}{\partial x_1} - \frac{(|\psi_0|^2)'}{|\psi_0|^2} \tag{13a}$$

$$A_{\rm i} = -\frac{\partial}{\partial p_2} - 2\alpha' \tag{13b}$$

$$A_{\rm r}^{+} = -\frac{\partial}{\partial x_1} - \frac{(|\psi_0|^2)'}{|\psi_0|^2} \tag{13c}$$

$$A_{\rm i}^{+} = \frac{\partial}{\partial p_2} - 2\alpha' \tag{13d}$$

and accordingly compute the possible bilinear forms, namely

$$A_{\rm r}^{+}A_{\rm r} = -\frac{\partial^2}{\partial x_1^2} + \frac{(|\psi_0|^2)''}{|\psi_0|^2} \tag{14a}$$

$$A_{\rm i}^{+}A_{\rm i} = -\frac{\partial^2}{\partial p_2^2} + 4\alpha'^2 - \frac{(|\psi_0|^2)''}{|\psi_0|^2} + \left\{\frac{(|\psi_0|^2)'}{|\psi_0|^2}\right\}^2 \tag{14b}$$

$$A_{\rm r}^{+}A_{\rm i} = \frac{\partial^2}{\partial x_1 \partial p_2} + 2\alpha'\frac{\partial}{\partial x_1} + 2\alpha'' + \frac{(|\psi_0|^2)'}{|\psi_0|^2}\frac{\partial}{\partial p_2} + 2\alpha' + 2\alpha'\frac{(|\psi_0|^2)'}{|\psi_0|^2} \tag{14c}$$

$$A_{\rm i}^{+}A_{\rm r} = \frac{\partial^2}{\partial x_1 \partial p_2} - 2\alpha'\frac{\partial}{\partial x_1} - \frac{(|\psi_0|^2)'}{|\psi_0|^2}\frac{\partial}{\partial p_2} + 2\alpha'' + 2\alpha'\frac{(|\psi_0|^2)'}{|\psi_0|^2} \tag{14d}$$

$$A_{\rm r}A_{\rm r}^{+} = -\frac{\partial^2}{\partial x_1^2} - \frac{(|\psi_0|^2)''}{|\psi_0|^2} + 2\left\{\frac{(|\psi_0|^2)'}{|\psi_0|^2}\right\}^2 \tag{14e}$$

$$A_{\rm i}A_{\rm i}^{+} = -\frac{\partial^2}{\partial p_2^2} - \frac{(|\psi_0|^2)''}{|\psi_0|^2} - \left\{\frac{(|\psi_0|^2)'}{|\psi_0|^2}\right\}^2 + 4\alpha'^2 \tag{14f}$$

$$A_{\rm i}A_{\rm r}^{+} = \frac{\partial^2}{\partial x_1 \partial p_2} + 2\alpha'\frac{\partial}{\partial x_1} + \frac{(|\psi_0|^2)'}{|\psi_0|^2}\frac{\partial}{\partial p_2} - 2\alpha'' + 2\alpha'\frac{(|\psi_0|^2)'}{|\psi_0|^2} \tag{14g}$$

$$A_{\rm r}A_{\rm i}^{+} = \frac{\partial^2}{\partial x_1 \partial p_2} - 2\alpha'\frac{\partial}{\partial x_1} - \frac{(|\psi_0|^2)'}{|\psi_0|^2}\frac{\partial}{\partial p_2} - 2\alpha'' + 2\alpha'\frac{(|\psi_0|^2)'}{|\psi_0|^2}. \tag{14h}$$

It is interesting to note that four out of the above eight bilinear forms of the operators $A_{\rm r}$, $A_{\rm r}^{+}$, $A_{\rm i}$, $A_{\rm i}^{+}$ can be readily identified with the Hamiltonians $H_{\rm r}^{(-)}$ and $H_{\rm i}^{(-)}$ as

$$A_{\rm r}^{+}A_{\rm r} - A_{\rm i}^{+}A_{\rm i} = H_{\rm r}^{(-)} \tag{15a}$$

$$A_{\rm r}^{+}A_{\rm i} + A_{\rm i}^{+}A_{\rm r} = H_{\rm i}^{(-)} \tag{15b}$$

whereas the remaining ones can be used to define $H_{\rm r}^{(+)}$ and $H_{\rm i}^{(+)}$ rather uniquely as

$$\begin{aligned} A_{\rm r}A_{\rm r}^{+} - A_{\rm i}A_{\rm i}^{+} &= H_{\rm r}^{(+)} = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial p_2^2} + V_{\rm r}^{(+)} \\ A_{\rm i}A_{\rm r}^{+} + A_{\rm r}A_{\rm i}^{+} &= H_{\rm i}^{(+)} = \frac{2\partial^2}{\partial x_1 \partial p_2} + V_{\rm i}^{(+)}. \end{aligned} \tag{16}$$

Here $V_{\rm r}^{(+)}$ and $V_{\rm i}^{(+)}$ are supersymmetric partner potentials to $V_{\rm r}^{(-)}$ and $V_{\rm i}^{(-)}$ respectively, which acquire the forms

$$V_{\rm r}^{(+)} = -V_{\rm r}^{(-)} + 2\left\{\frac{(|\psi_0|^2)'}{|\psi_0|^2}\right\}^2 - 8\alpha'^2 \qquad V_{\rm i}^{(+)} = -V_{\rm i}^{(-)} + 8\alpha'\frac{(|\psi_0|^2)'}{|\psi_0|^2}. \tag{17}$$

In SUSY QM, since the ground-state eigenfunction ψ_0 is related to the superpotential $W(x)$ via the Reccati transformation, namely

$$W(x) = -\frac{\psi_0'}{\psi_0} \qquad \left(\psi_0' = \frac{{\rm d}\psi_0}{{\rm d}x}\right)$$

we recast the above results in terms of the real and imaginary parts of the superpotential $W_{\rm r}(x_1, p_2)$ and $W_{\rm i}(x_1, p_2)$ in the complex phase space. Using the definitions of equation (6), $W_{\rm r}$ and $W_{\rm i}$ can now be expressed in terms of $|\psi_0|$ and α as

$$W_{\rm r} = -\frac{(|\psi_0|^2)'}{|\psi_0|^2} \qquad W_{\rm i} = -2\alpha'. \tag{18}$$

In terms of $W_{\rm r}$ and $W_{\rm i}$, equations (13*a*)–(13*d*) can be written as

$$A_{\rm r} = \frac{\partial}{\partial x_1} + W_{\rm r} \qquad A_{\rm i} = -\frac{\partial}{\partial p_2} + W_{\rm i} \qquad A_{\rm r}^+ = -\frac{\partial}{\partial x_1} + W_{\rm r} \qquad A_{\rm i}^+ = \frac{\partial}{\partial p_2} + W_{\rm i} \tag{19}$$

and the potential functions (9) and (17) now become

$$V_{\rm r}^{(-)} = W_{\rm r}^2 - W_{\rm i}^2 - 2\frac{\partial W_{\rm r}}{\partial x_1} \tag{20a}$$

$$V_{\rm i}^{(-)} = 2W_{\rm r}W_{\rm i} - 2\frac{\partial W_{\rm i}}{\partial x_1} \tag{20b}$$

$$V_{\rm r}^{(+)} = 2\left(W_{\rm r}^2 - W_{\rm i}^2\right) - V_{\rm r}^{(-)} \tag{20c}$$

$$V_{\rm i}^{(+)} = 4W_{\rm r}W_{\rm i} - V_{\rm i}^{(-)}. \tag{20d}$$

One can immediately note from equations (20*a*)–(20*d*) that the supersymmetric potentials $V^{(\pm)}$, now given by $V^{(\pm)} = V_{\rm r}^{(\pm)} \pm {\rm i}V_{\rm i}^{(\pm)} = W^2 \pm \frac{{\rm d}W}{{\rm d}x}(W = W_{\rm r} + {\rm i}W_{\rm i})$, indeed have the same structure as for the case of real SUSY QM. Another notable feature of the present method, which conforms to that of real SUSY QM, is the relation

$$\tfrac{1}{2}[V^{(+)} + V^{(-)}] = W^2$$

i.e., the square of the superpotential is the average of the complex supersymmetric partner potentials $V^{(+)}$ and $V^{(-)}$. Interestingly, the result $[A, A^+] = W'$ of real SUSY QM is now translated into the following four commutation relations:

$$\left[A_{\rm r}, A_{\rm r}^+\right] = 2W_{\rm r}' \qquad \left[A_{\rm i}, A_{\rm i}^+\right] = -2W_{\rm r}' \qquad \left[A_{\rm r}, A_{\rm i}^+\right] = 2W_{\rm i}' \qquad \left[A_{\rm i}, A_{\rm r}^+\right] = 2W_{\rm r}' \tag{21}$$

from which one can easily verify that

$$\left[A_{\rm r}, A_{\rm r}^+\right] + \left[A_{\rm i}, A_{\rm i}^+\right] = 0 \tag{22a}$$

$$\left[A_{\rm r}, A_{\rm i}^+\right] = \left[A_{\rm i}, A_{\rm r}^+\right] = 2W_{\rm i}'. \tag{22b}$$

From these results it can be noted that $W_{\rm r}'$ is proportional to the commutator of $A_{\rm r}$ and $A_{\rm r}^+$ or of $A_{\rm i}$ and $A_{\rm i}^+$, whereas $W_{\rm i}'$ is proportional to the commutator of $A_{\rm r}$ and $A_{\rm i}^+$ or of $A_{\rm i}$ and $A_{\rm r}^+$.

3.2. Relationship between the eigenvalues and eigenfunctions of $H^{(+)}$ and $H^{(-)}$

Unlike the real SUSY QM where one deals with two real Hamiltonians $H^{(+)}$ and $H^{(-)}$, here in the complex formulation of SUSY QM, we have instead four real Hamiltonians $H_{\rm r}^{(+)}$, $H_{\rm i}^{(+)}$, $H_{\rm r}^{(-)}$ and $H_{\rm i}^{(-)}$. If one assumes that each of these Hamiltonians has its own set of independent eigenfunctions, then

$$\begin{aligned} H_{\rm r}^{(+)}\psi_{n{\rm r}}^{(+)} &= E_{n{\rm r}}^{(+)}\psi_{n{\rm r}}^{(+)} \qquad & H_{\rm i}^{(+)}\psi_{n{\rm i}}^{(+)} &= E_{n{\rm i}}^{(+)}\psi_{n{\rm i}}^{(+)} \\ H_{\rm r}^{(-)}\psi_{n{\rm r}}^{(-)} &= E_{n{\rm r}}^{(-)}\psi_{n{\rm r}}^{(-)} \qquad & H_{\rm i}^{(-)}\psi_{n{\rm i}}^{(-)} &= E_{n{\rm i}}^{(-)}\psi_{n{\rm i}}^{(-)} \end{aligned} \tag{22}$$

where $n = 0, 1, 2, \ldots$ represents the number of nodes in the corresponding eigenfunction in the above eigenvalue equations.

Next, using the definitions in equations (15) and (16), we compute the trilinear forms involving the real and imaginary parts of A and A^{+}. This yields the following results:

$$\begin{aligned} H_{\rm r}^{(+)}\big(A_{\rm r}\psi_{n{\rm r}}^{(-)}\big) &= E_{n{\rm r}}^{(-)}\big(A_{\rm r}\psi_{n{\rm r}}^{(-)}\big) \qquad & H_{\rm i}^{(+)}\big(A_{\rm i}\psi_{n{\rm r}}^{(-)}\big) &= E_{n{\rm i}}^{(-)}\big(A_{\rm i}\psi_{n{\rm r}}^{(-)}\big) \\ H_{\rm r}^{(+)}\big(A_{\rm i}\psi_{n{\rm i}}^{(-)}\big) &= E_{n{\rm r}}^{(-)}\big(A_{\rm i}\psi_{n{\rm r}}^{(-)}\big) \qquad & H_{\rm i}^{(+)}\big(A_{\rm r}\psi_{n{\rm i}}^{(+)}\big) &= E_{n{\rm i}}^{(-)}\big(A_{\rm r}\psi_{n{\rm i}}^{(-)}\big) \end{aligned} \tag{23}$$

and

$$\begin{aligned} H_{\rm r}^{(-)}\big(A_{\rm r}^{+}\psi_{n{\rm r}}^{(+)}\big) &= E_{n{\rm r}}^{(+)}\big(A_{\rm r}^{+}\psi_{n{\rm r}}^{(+)}\big) \qquad & H_{\rm i}^{(-)}\big(A_{\rm i}^{+}\psi_{n{\rm r}}^{(+)}\big) &= E_{n{\rm i}}^{(+)}\big(A_{\rm i}^{+}\psi_{n{\rm i}}^{(+)}\big) \\ H_{\rm i}^{(-)}\big(A_{\rm r}^{+}\psi_{n{\rm r}}^{(+)}\big) &= E_{n{\rm i}}^{(+)}\big(A_{\rm i}^{+}\psi_{n{\rm r}}^{(+)}\big) \qquad & H_{\rm r}^{(-)}\big(A_{\rm i}^{+}\psi_{n{\rm r}}^{(+)}\big) &= E_{n{\rm r}}^{(+)}\big(A_{\rm i}^{+}\psi_{n{\rm r}}^{(+)}\big). \end{aligned} \tag{24}$$

The above results clearly indicate that $E_{n{\rm r}}^{(-)}$, $E_{n{\rm i}}^{(-)}$, $E_{n{\rm r}}^{(+)}$ and $E_{n{\rm i}}^{(+)}$ are the eigenvalues of the Hamiltonians $H_{\rm r}^{(+)}$, $H_{\rm i}^{(+)}$, $H_{\rm r}^{(-)}$ and $H_{\rm i}^{(-)}$ with the corresponding (rather new compared to equations (22)) eigenfunctions $\big(A_{\rm r}\psi_{n{\rm r}}^{(-)}\big)$, $\big(A_{\rm i}\psi_{n{\rm r}}^{(-)}\big)$, $\big(A_{\rm r}^{+}\psi_{n{\rm r}}^{(+)}\big)$ and $\big(A_{\rm i}^{+}\psi_{n{\rm i}}^{(+)}\big)$. This immediately implies that

$$E_{n{\rm r}}^{(+)} = E_{(n+1){\rm r}}^{(-)} \qquad \psi_{n{\rm r}}^{(+)} = \big[E_{(n+1){\rm r}}^{(-)}\big]^{1/2}\big(A_{\rm r}\psi_{(n+1){\rm r}}^{(-)}\big) \tag{25}$$

$$E_{n{\rm i}}^{(+)} = E_{(n+1){\rm i}}^{(-)} \qquad \psi_{n{\rm i}}^{(+)} = \big[E_{(n+1){\rm i}}^{(-)}\big]^{1/2}\big(A_{\rm i}\psi_{(n+1){\rm i}}^{(-)}\big). \tag{26}$$

4. Shape-invariant complex potentials

In the real domain, the concept of shape-invariant potentials, as enumerated by Gendenshtein [37], describes the underlying symmetry of a variety of known potentials which admit explicit solution to the problem in nonrelativistic quantum mechanics. If the pair of SUSY partner potentials in the real domain, i.e., $V^{(\pm)}$ are similar in shape but differ only in the parameters appearing in the potential, then they are said to be shape invariant. More precisely, if a_1 represents the set of parameters appearing in the potential and a_2 is a function of a_1 (say $a_2 = f(a_1)$), then the shape-invariance property of the potential will require that

$$V^{(+)}(x; a_1) = V^{(-)}(x; a_2) + R(a_1) \tag{27}$$

where the remainder $R(a_1)$ is a function of a_1 and is independent of x. Such SUSY partner potentials $V^{(\pm)}$ are designated as shape invariant.

However, the situation is different in the case of complex potentials. The above definition of shape invariance needs to be generalized with respect to the real and imaginary parts of $V(x)$. In fact, in what follows, we proceed with the idea that for complex potentials, the real and imaginary parts of the SUSY partner potentials $V_{\rm r}^{(\pm)}$ and $V_{\rm i}^{(\pm)}$ should individually possess the property of shape invariance, i.e., these parts separately should have similar shapes but differ only in the parameters appearing in the potential. Since we are considering here the complexity of the potential parameters also, the defining equation (27) will involve their real and imaginary parts as well. Thus, if $a_{\rm r}$ and $a_{\rm i}$ are the sets representing the real and imaginary parts of parameters appearing in the potential $V_{\rm r}^{(\pm)}$ and $V_{\rm i}^{(\pm)}$, then equation (27) can be transcribed in terms of the relations,

$$\begin{aligned} V_{\rm r}^{(+)}(x_1, p_2, a_{\rm r}, a_{\rm i}) &= V_{\rm r}^{(-)}(x_1, p_2, a_{\rm r}', a_{\rm i}') + R_1(a_{\rm r}, a_{\rm i}) \\ V_{\rm i}^{(+)}(x_1, p_2, a_{\rm r}, a_{\rm i}) &= V_{\rm i}^{(-)}(x_1, p_2, a_{\rm r}', a_{\rm i}') + R_2(a_{\rm r}, a_{\rm i}) \end{aligned} \tag{28}$$

where $a_{\rm r}'$ and $a_{\rm i}'$ respectively represent functions of $a_{\rm r}$ and $a_{\rm i}$ and the remainders R_1 and R_2 are functions of $a_{\rm r}$ and $a_{\rm i}$ alone and are independent of phase space variables x_1 and p_2.

In the following section, we present a pedagogical analysis of a variety of complex potentials in the light of the above generalized concept of shape invariance.

5. Analysis of some complex potentials

In order to lend additional support to our method of formulation of complex SUSY QM, we investigate here the complex version of some celebrated potentials admitting shape invariance in the real domain. For this purpose, we make use of equation (28) and try to get an insight into the shape-invariance property of complex potentials.

5.1. Complex harmonic oscillator

We first consider the simplest example of a complex harmonic oscillator of the form

$$V(x) = ax^2 \pm \sqrt{a} \qquad a = a_{\rm r} + {\rm i}a_{\rm i}. \tag{29a}$$

The superpotential corresponding to this potential from equation (4) is given by

$$W(x) = \sqrt{a}x. \tag{29b}$$

The real and imaginary parts of (29*b*) take the forms

$$\begin{aligned} W_{\rm r} &= \left(a_{\rm r}^2 + a_{\rm i}^2\right)^{1/4}\left(x_1\cos\frac{\theta}{2} - p_2\sin\frac{\theta}{2}\right) \\ W_{\rm i} &= \left(a_{\rm r}^2 + a_{\rm i}^2\right)^{1/4}\left(x_1\sin\frac{\theta}{2} + p_2\cos\frac{\theta}{2}\right) \end{aligned} \tag{30}$$

where $\theta = \tan^{-1}\frac{a_{\rm i}}{a_{\rm r}}$. These results, when used in equations (20*a*)–(20*d*), lead to the real and imaginary parts of the SUSY partner potentials as

$$\begin{aligned} V_{\rm r}^{(+)} &= a_{\rm r}\left(x_1^2 - p_2^2\right) - 2a_{\rm i}x_1p_2 \pm \tfrac{1}{\sqrt{2}}(|a| \pm a_{\rm r})^{1/2} \\ V_{\rm r}^{(-)} &= a_{\rm r}\left(x_1^2 - p_2^2\right) - 2a_{\rm i}x_1p_2 \mp \tfrac{1}{\sqrt{2}}(|a| \pm a_{\rm r})^{1/2} \\ V_{\rm i}^{(+)} &= 2a_{\rm r}x_1p_2 + a_{\rm i}\left(x_1^2 - p_2^2\right) \pm \tfrac{1}{\sqrt{2}}(|a| \mp a_{\rm r})^{1/2} \\ V_{\rm i}^{(-)} &= 2a_{\rm r}x_1p_2 + a_{\rm i}\left(x_1^2 - p_2^2\right) \mp \tfrac{1}{\sqrt{2}}(|a| \mp a_{\rm r})^{1/2} \end{aligned} \tag{31}$$

where we have used

$$\begin{aligned} \cos\frac{\theta}{2} &= \pm[(|a| + a_{\rm r})/2|a|]^{1/2} \\ \sin\frac{\theta}{2} &= \pm[(|a| - a_{\rm r})/2|a|]^{1/2} \qquad \text{with} \quad |a| = \left(a_r^2 + a_{\rm i}^2\right)^{1/2}. \end{aligned}$$

It is not difficult to verify that these forms of SUSY partner potentials conform to the shape-invariance conditions (28) with the result

$$R_1(a_{\rm r}, a_{\rm i}) = \sqrt{2}(|a| \pm a_{\rm r})^{1/2} \qquad \text{and} \qquad R_2(a_{\rm r}, a_{\rm i}) = \sqrt{2}(|a| \mp a_{\rm r})^{1/2}.$$

Further, the ground-state solution (2*b*) for the potential (29*a*) is given by

$$\psi_0(x) = \exp\left[\mp\frac{1}{2\sqrt{2}}\{(|a| + a_{\rm r})^{1/2} + (|a| - a_{\rm r})^{1/2}\}x^2\right]. \tag{32}$$

5.2. Complex harmonic potential with an inverse harmonic term

Now we consider the complex potential

$$V(x) = ax^2 + \frac{b}{x^2} \qquad (a, b \text{ complex}) \tag{33a}$$

for which one immediately identifies the superpotential as

$$W(x) = \sqrt{a}x + \frac{\sqrt{b}}{x}. \tag{33b}$$

Proceeding as before, the real and imaginary parts of the SUSY partner potentials now take the forms

$$\begin{aligned} V_{\mathrm{r}}^{(\pm)} = {}& \varepsilon\left(a_{\mathrm{r}} + \frac{b_{\mathrm{r}}}{|x|^4}\right)\left(x_1^2 - p_2^2\right) - 2a_{\mathrm{i}}x_1p_2 + \{(|a| + \varepsilon a_{\mathrm{r}})(|b| + \varepsilon b_{\mathrm{r}})\}^{1/2} \\ & - \{(|a| - \varepsilon a_{\mathrm{r}})(|b| - \varepsilon b_{\mathrm{r}})\}^{1/2} + \sqrt{2}\varepsilon\left\{\pm(|a| + \varepsilon a_{\mathrm{r}})^{1/2} \mp \frac{2}{|x|^4}(|b| - \varepsilon b_{\mathrm{r}})^{1/2}x_1p_2\right. \\ & \left.\mp \frac{1}{|x|^4}(|b| + \varepsilon b_{\mathrm{r}})^{1/2}\left(x_1^2 - p_2^2\right)\right\} \end{aligned} \tag{34a}$$

$$\begin{aligned} V_{\mathrm{i}}^{(\pm)} = {}& \left(a_{\mathrm{i}} + \frac{b_{\mathrm{i}}}{|x|^4}\right)\left(x_1^2 - p_2^2\right) + 2\varepsilon(a_{\mathrm{r}} - b_{\mathrm{r}})x_1p_2 + 2(|ab| - a_{\mathrm{r}}b_{\mathrm{r}}) \\ & + \sqrt{2}\left\{\pm(|a| - \varepsilon a_{\mathrm{r}})^{1/2} - \frac{1}{|x|^4}(|b| - \varepsilon b_{\mathrm{r}})\left(x_1^2 - p_2^2\right) \pm \frac{2}{|x|^4}(|b| + \varepsilon b_{\mathrm{r}})^{1/2}x_1p_2\right\} \end{aligned} \tag{34b}$$

where $\varepsilon = \pm 1$ for all cases. Comparison of $V^{(+)}$ and $V^{(-)}$ in equations (34*a*) and (34*b*) immediately leads to the following relations among the SUSY partner potentials

$$\begin{aligned} V_{\mathrm{r}}^{(+)}(x_1, p_2, |a| \pm a_{\mathrm{r}}, |b| \pm b_{\mathrm{r}}) &= V_{\mathrm{r}}^{(-)}(x_1, p_2, -(|a| \pm a_{\mathrm{r}}), -(|b| \pm b_{\mathrm{r}})) \\ V_{\mathrm{i}}^{(+)}(x_1, p_2, |a| \mp a_{\mathrm{r}}, |b| \mp b_{\mathrm{r}}) &= V_{\mathrm{i}}^{(-)}(x_1, p_2, -(|a| \mp a_{\mathrm{r}}), -(|b| \mp b_{\mathrm{r}})) \end{aligned} \tag{35}$$

which again conform to the shape-invariance condition (28). In this case, the ground-state solution for the potential (33*a*) is given by

$$\psi_0(x) = x^{-\sqrt{b}} \exp\left[\mp\frac{1}{2\sqrt{2}}\{(|a| + a_{\mathrm{r}})^{1/2} + (|a| - a_{\mathrm{r}})^{1/2}\}x^2\right].$$

5.3. *Complex-exponential potentials*

The exponential potentials in the real domain of x have been a subject of considerable interest due to their applications in numerous physical problems. Recently, the complex form of such potentials has been studied by several authors [11, 18, 19, 22, 32, 33]. We wish to analyse here such potentials in complex phase space characterized by (1) in the context of their shape invariance. In particular, we study the Morse potential and a generalized form of the exponential potentials discussed recently by Jia *et al* [22], of which some cases such as Rosen–Morse potential and the Scarf II potential will appear as special limiting cases.

5.3.1. Complex Morse potential. Consider the Morse potential written in one dimension [38] as

$$V(x) = A^2 + B^2\,\mathrm{e}^{-2\alpha x} - 2B\left(A - \frac{\alpha}{2}\right)\mathrm{e}^{-\alpha x}$$

where A, B and α are assumed to be complex. The superpotential corresponding to the above form can be written as

$$W(x) = A - B\,\mathrm{e}^{-\alpha x}$$

which, in turn, yields the real and imaginary parts of the SUSY partner potentials as

$$V_{\mathrm{r}}^{(\pm)} = \left(A_{\mathrm{r}}^2 - A_{\mathrm{i}}^2\right) + \mathrm{e}^{-2y}\left[\left(B_{\mathrm{r}}^2 - B_{\mathrm{i}}^2\right)\cos 2z - 2B_{\mathrm{r}}B_{\mathrm{i}}\sin 2z\right] + 2\,\mathrm{e}^{-y}\{(A_{\mathrm{r}}B_{\mathrm{i}} + A_{\mathrm{i}}B_{\mathrm{r}} \mp \alpha_{\mathrm{r}}B_{\mathrm{i}} \pm \alpha_{\mathrm{i}}B_{\mathrm{r}})\sin z + (A_{\mathrm{i}}B_{\mathrm{i}} - A_{\mathrm{r}}B_{\mathrm{r}} \pm \alpha_{\mathrm{r}}B_{\mathrm{r}} \mp \alpha_{\mathrm{i}}B_{\mathrm{i}})\cos z\} \tag{35a}$$

$$V_{\mathrm{i}}^{(\pm)} = 2A_{\mathrm{r}}A_{\mathrm{i}} + \mathrm{e}^{-2y}\left[\left(B_{\mathrm{i}}^2 - B_{\mathrm{r}}^2\right)\sin 2z - 2B_{\mathrm{r}}B_{\mathrm{i}}\cos 2z\right] + 2\,\mathrm{e}^{-y}[(A_{\mathrm{i}}B_{\mathrm{i}} - A_{\mathrm{r}}B_{\mathrm{r}} \mp \alpha_{\mathrm{r}}B_{\mathrm{r}} \pm \alpha_{\mathrm{i}}B_{\mathrm{i}})\sin z + (\pm\alpha_{\mathrm{r}}B_{\mathrm{i}} \mp \alpha_{\mathrm{i}}B_{\mathrm{r}} - A_{\mathrm{r}}B_{\mathrm{i}} - A_{\mathrm{i}}B_{\mathrm{r}})\cos z] \tag{35b}$$

where

$$y = \alpha_{\mathrm{r}}x_1 - \alpha_{\mathrm{i}}p_2 \qquad z = \alpha_{\mathrm{i}}x_1 + \alpha_{\mathrm{r}}p_2. \tag{36}$$

From equations (35*a*), (35*b*), it can be immediately noted that $V_{\mathrm{r}}^{(+)}$ and $V_{\mathrm{r}}^{(-)}$ are now related as

$$V_{\mathrm{r}}^{(+)}(x_1, p_2, A_{\mathrm{r}}, A_{\mathrm{i}}, B_{\mathrm{r}}, B_{\mathrm{i}}) = V_{\mathrm{r}}^{(-)}(x_1, p_2, -A_{\mathrm{r}}, -A_{\mathrm{i}}, -B_{\mathrm{r}}, -B_{\mathrm{i}}) \tag{37a}$$

whereas the corresponding relation between $V_{\mathrm{i}}^{(+)}$ and $V_{\mathrm{i}}^{(-)}$ is

$$V_{\mathrm{i}}^{(+)}(x_1, p_2, A_{\mathrm{r}}, A_{\mathrm{i}}, B_{\mathrm{r}}, B_{\mathrm{i}}) = V_{\mathrm{i}}^{(-)}(x_1, p_2, -A_{\mathrm{r}}, -A_{\mathrm{i}}, -B_{\mathrm{r}}, -B_{\mathrm{i}}). \tag{37b}$$

Further note that these relations conform to equation (28) with no remainders present. This implies that the complex Morse potential in one dimension is shape invariant. Clearly, the ground-state eigenfunction from (2*b*) can be written as

$$\psi_0(x) = \exp\left[-Ax + \frac{B}{\alpha}\,\mathrm{e}^{-\alpha x}\right].$$

5.3.2. A generalized complex-exponential potential. Following the work of Jia *et al* [22] we now consider a generalized version of the complex-exponential potentials described by

$$V(x) = Q_1^2 + \frac{2Q_1(Q_2 + Q_3\,\mathrm{e}^{\alpha x})}{(\mathrm{e}^{2\alpha x} + q)} + \frac{\left[Q_2^2 + (2Q_2Q_3 + Q_3\alpha)\,\mathrm{e}^{\alpha x} + \left(Q_3^2 - 2Q_2\alpha\right)\mathrm{e}^{2\alpha x} - Q_3\alpha\,\mathrm{e}^{3\alpha x}\right]}{(\mathrm{e}^{2\alpha x} + q)^2} \tag{38}$$

which, for the case when $Q_1 = A$, $Q_2 = 0$, $Q_3 = -B$, $q = 0$, reduces to the Morse form (cf subsection 5.3.1). Note that here we consider Q_i $(i = 1, 2, 3)$, q and α as complex.

The corresponsing superpotential now becomes [22]

$$W(x) = Q_1 + \frac{Q_2}{\mathrm{e}^{2\alpha x} + q} + \frac{Q_3\,\mathrm{e}^{\alpha x}}{\mathrm{e}^{2\alpha x} + q} \tag{39}$$

with real and imaginary parts written in compact form as

$$W_{\mathrm{r}} = Q_{1r} + \frac{1}{(M^2 + N^2)}(MR + NS) \tag{40a}$$

$$W_{\mathrm{i}} = Q_{1i} + \frac{1}{(M^2 + N^2)}(MS - NR) \tag{40b}$$

where

$$M = \mathrm{e}^{2y}\cos 2z + q_{\mathrm{r}} \qquad N = \mathrm{e}^{2y}\sin 2z + q_{\mathrm{i}}$$
$$R = Q_{2\mathrm{r}} + Q_{3\mathrm{r}}\,\mathrm{e}^{y}\cos z - Q_{3\mathrm{i}}\,\mathrm{e}^{y}\sin z$$
$$S = Q_{2\mathrm{i}} + Q_{3\mathrm{i}}\,\mathrm{e}^{y}\cos z + Q_{3\mathrm{r}}\,\mathrm{e}^{y}\sin z$$

and

$$y = \alpha_r x_1 - \alpha_i p_2 \qquad z = \alpha_i x_1 + \alpha_r p_2.$$

Using these forms of W_r and W_i in equations (20*a*) and (20*b*), one obtains the expressions for the superpartner potentials as

$$\begin{aligned}V_r^{(\pm)} = EF + &\frac{1}{(M^2+N^2)^2}[\{M(A + B\,e^y\cos z + D\,e^y\sin z) - N(C + D\,e^y\cos z \\ &- B\,e^y\sin z)\}\{M(C + D\,e^y\cos z - B\,e^y\sin z) + N(A + B\,e^y\cos z \\ &+ D\,e^y\sin z)\} \mp 4\,e^{2y}\{M(\alpha_r\cos 2z - \alpha_i\sin 2z) + N(\alpha_i\cos 2z + \alpha_r\sin 2z)\} \\ &\times\{M((A+C) + (B+D)\,e^y\cos z - (B-D)\,e^y\sin z) + N((A-C) \\ &+ (B-D)\,e^y\cos z + (B+D)\,e^y\sin z)\}] + \frac{1}{2(M^2+N^2)}[M\{(AF+EC) \\ &+ (BF+DE)\,e^y\cos z - (BE-DF)\,e^y\sin z \pm (B+D)\,e^y(\alpha_r\cos z \\ &- \alpha_i\sin z) \mp (B-D)\,e^y(\alpha_r\sin z + \alpha_i\cos z)\} + N\{(AE-CF) \\ &+ (BE-DF)\,e^y\cos z + (BF+DE)\,e^y\sin z \pm (B-D)\,e^y(\alpha_r\cos z - \alpha_i\sin z) \\ &\pm (B+D)\,e^y(\alpha_r\sin z + \alpha_i\cos z)\} \pm 2\,e^{2y}(\alpha_r\cos 2z - \alpha_i\sin 2z)\{(A+C) \\ &+ (B+D)\,e^y\cos z - (B-D)\,e^y\sin z\} \pm 2\,e^{2y}(\alpha_i\cos 2z + \alpha_r\sin 2z)\{(A-C) \\ &+ (B-D)\,e^y\cos z + (B+D)\,e^y\sin z\}] \end{aligned} \tag{41a}$$

$$\begin{aligned}V_i^{\pm} = \frac{1}{2}(E^2 - F^2) + &\frac{1}{(M^2+N^2)^2}\left[\frac{1}{2}(M^2-N^2)\{(A+C) + (B+D)\,e^y\cos z\right. \\ &- (B-D)\,e^y\sin z\}\{(A-C) + (B-D)\,e^y\cos z + (B+D)\,e^y\sin z\} \\ &- 2MN(A + B\,e^y\cos z + D\,e^y\sin z)(C + D\,e^y\cos z - B\,e^y\sin z) \\ &\mp 4\,e^{2y}\{M(\alpha_r\cos 2z - \alpha_i\sin 2z) + N(\alpha_i\cos 2z + \alpha_r\sin 2z)\}\{M((A-C) \\ &+ (B-D)\,e^y\cos z - (B+D)\,e^y\sin z)\} - N((A+C) + (B+D)\,e^y\cos z \\ &\left.- (B-D)\,e^y\sin z)\}\right] + \frac{1}{(M^2+N^2)}[M\{(AE-CF) + (BE-DF)\,e^y\cos z \\ &+ (BF+DE)\,e^y\sin z \pm (B-D)\,e^y(\alpha_r\cos z - \alpha_i\sin z) \\ &\pm (B+D)\,e^y(\alpha_r\sin z + \alpha_i\cos z)\} + N\{(AF+CE) + (BF+DE)\,e^y\cos z \\ &- (BE-DF)\,e^y\sin z \mp (B+D)\,e^y(\alpha_r\cos z - \alpha_i\sin z) \\ &\pm (B-D)\,e^y(\alpha_r\sin z + \alpha_i\cos z)\} \pm 2\,e^{2y}\cos 2z - \alpha_i\sin 2z)\{(A-C) \\ &+ (B-D)\,e^y\cos z + (B+D)\,e^y\sin z\} \mp 2\,e^{2y}(\alpha_i\cos 2z + \alpha_r\sin 2z) \\ &\times\{(A+C) + (B+D)\,e^y\cos z - (B-D)\,e^y\sin z\}] \end{aligned} \tag{41b}$$

where

$$A = Q_{2r} + Q_{2i} \qquad B = Q_{3r} + Q_{3i} \qquad C = Q_{2r} - Q_{2i}$$
$$D = Q_{3r} - Q_{3i} \qquad E = Q_{1r} + Q_{1i} \qquad F = Q_{1r} - Q_{1i}.$$

Next we list some special cases of the above general results for the complex-exponential potentials.

(i) *Rosen–Morse potential*: Note that for $q = 1$, $Q_3 = 0$ implying $B = D = 0$, one immediately obtains the results for the complex Rosen–Morse potential of the type

$$V(x) = Q_2^2 + (Q_1 + Q_2)^2 - 2Q_2(Q_1 + Q_2)\tanh\alpha x - Q_2\left(Q_2 - \frac{\alpha}{\sqrt{2}}\right)\operatorname{sech}^2\alpha x$$

which corresponds to the superpotential

$$W = Q_1 + Q_2(1 - \tanh \alpha x).$$

For this case the comparison of various results in equations (41) immediately yields the following reflection properties:

$$\begin{aligned} V_{\rm r}^{(+)}(x_1, p_2, Q_{1\rm r}, Q_{1\rm i}, Q_{2\rm r}, Q_{2\rm i}) &= V_{\rm r}^{(-)}(x_1, p_2, Q_{1\rm r}, Q_{1\rm i}, -Q_{2\rm r}, -Q_{2\rm i}) \\ V_{\rm i}^{(+)}(x_1, p_2, Q_{1\rm r}, Q_{1\rm i}, Q_{2\rm r}, Q_{2\rm i}) &= V_{\rm i}^{(-)}(x_1, p_2, Q_{1\rm r}, Q_{1\rm i}, -Q_{2\rm r}, -Q_{2\rm i}). \end{aligned} \tag{42}$$

(ii) *Scarf II potential*: For the choice $q = 1$, $Q_2 = -2Q_1$, implying $A = -2E$, $C = -2F$, one obtains the Scarf II potential from (38) as

$$V(x) = Q_1^2 + \tfrac{1}{4}\big(Q_3^2 - 4Q_1^2 - 4\alpha Q_1\big)\,\mathrm{sech}^2\,\alpha x - \big(Q_1 + \tfrac{1}{2}\alpha\big)Q_3\,\mathrm{sech}\,\alpha x \tanh \alpha x$$

with the corresponding superpotential

$$W(x) = \tfrac{1}{2}Q_3\,\mathrm{sech}\,\alpha x - Q_1 \tanh \alpha x.$$

It is straightforward to verify that for this case the results in equations (41) satisfy the following reflection properties:

$$\begin{aligned} V_{\rm r}^{(+)}(x_1, p_2, Q_{1\rm r}, Q_{1\rm i}, Q_{3\rm r}, Q_{3\rm i}) &= V_{\rm r}^{(-)}(x_1, p_2, -Q_{1\rm r}, -Q_{1\rm i}, -Q_{3\rm r}, -Q_{3\rm i}) \\ V_{\rm i}^{(+)}(x_1, p_2, Q_{1\rm r}, Q_{1\rm i}, Q_{3\rm r}, Q_{3\rm i}) &= V_{\rm i}^{(-)}(x_1, p_2, -Q_{1\rm r}, -Q_{1\rm i}, -Q_{3\rm r}, -Q_{3\rm i}). \end{aligned} \tag{43}$$

It is truly remarkable that the parameters appearing in these potentials exhibit the reflection property (cf equations (35), (37), (42) and (43)) which otherwise is not explicitly manifest in conventional (real) SUSY QM. Some of these features of the Scarf II potential will be compared in the following section with those obtained recently by Levai and Znojil [36].

6. Connection with other approaches on SUSY QM

In this section, we briefly remark on the results derived in the present approach and those obtained using other variants of SUSY QM including the PT-symmetric ones. In particular, we demonstrate a possible linkage of the present work with the works of Das *et al* [3] and Amado *et al* [7], respectively with regard to the SUSY QM in higher (two) dimensions and the multiple copies of Hilbert space for a system. Also, the concept of 'quasi-parity' used by Levai and Znojil [36] for PT-symmetric Hamiltonians is analysed in the present framework.

Before proceeding further, one can immediately note the correspondence between the function $g(x)$ in the eigenvalue ansatz method for solving the analogous Schrödinger equation (cf [23, 24]) and the superpotential $W(x)$ (cf equation (2*b*) for the conventional SUSY QM and equation (18) with α' defined in equation (12) in the present approach). In general it can be seen that $g(x) = -\int^x W(y)\,\mathrm{d}y$.

In the present approach, the complex phase space is produced by introducing the imaginary parts p_2 and x_2 in the variables x and p (cf equation (1)), respectively. The use of the analyticity property of the (classical) Hamiltonian $H(x, p)$ has further revealed [31] some important properties such as the integrability of two, two-dimensional real Hamiltonian systems. In the same spirit, it is not difficult to see a possible connection of the present complex SUSY QM with the SUSY QM investigated by Das *et al* [3] in two real dimensions. In fact, there exists a reasonable correspondence between the supercharges derived in the present work (cf equations (19)) and the frame-independent constructions of Das *et al* [3] in two real dimensions.

In the work of Das *et al* [3], the interaction through a vector superpotential $\vec{W}$ is introduced by defining the supercharge at the free particle level as

$$A_{\mathrm{free}} = \vec{e}^{+} \cdot \vec{\nabla} \qquad (e^{+} = e_x + \mathrm{i}e_y) \tag{44}$$

leading finally to the supercharge in a frame-independent way as

$$A = \vec{e}^{+} \cdot (\vec{\nabla} + \vec{W}) \tag{45}$$

and the superpartner Hamiltonian as $H^{(-)} = A^{+}A$ and $H^{(+)} = AA^{+}$. It may be mentioned that while the present approach provides four distinct expressions for the supercharges (cf equations (19)) in a very general manner, two of them however coincide with the frame independent constructions of Das *et al*. In fact, if we define A_{free} in (44) as $A_{\mathrm{free}} = \vec{e}^{-} \cdot \vec{\nabla}$ and construct A as $A = \vec{e}^{-} \cdot (\vec{\nabla} + \vec{W})$, then it is easy to see that real and imaginary parts of A turn out to be the same as equations (19*a*) and (19*b*).

In order to describe the emission of isospectral γ-rays from the adjacent nuclei in the superdeformed region, Amado *et al* [7] have used the concept of pseudospin and the fact that in supersymmetry the total number of bosons plus fermions is conserved. This has led to the constructions of supermultiplet state and thereby connecting the spectra of four adjacent nuclei. In the present work, our constructions of $H_{\mathrm{r}}^{(+)}$, $H_{\mathrm{i}}^{(+)}$, $H_{\mathrm{r}}^{(-)}$ and $H_{\mathrm{i}}^{(-)}$ in equations (15) and (16) in conjunction with equations (22) to (26) will definitely give rise to similar features as far as the connection of spectra of four nuclei is concerned. In our case, however, the choice of nuclei is restricted in the sense that energy eigenvalues have the same pairing preferences (cf equations (25) and (26)).

Levai and Znojil [36], while investigating the relationship between PT-symmetry and SUSY QM, have recently observed that the PT-symmetric version of the Scarf II potential given by

$$V(x) = -\left[\left(\frac{\bar{\alpha}+\bar{\beta}}{2}\right) + \left(\frac{\bar{\alpha}-\bar{\beta}}{2}\right)^{2} - \frac{1}{4}\right] \mathrm{sech}^{2} x + 2\mathrm{i}\left(\frac{\bar{\alpha}+\bar{\beta}}{2}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{2}\right) \mathrm{sech}\, x \tanh x \tag{46}$$

has a broader range of normalizable states than the Hermitian version of the same potential. This observation was attributed to the fact that potential (46) is invariant under $\bar{\alpha} \to -\bar{\alpha}$ transformation. Also, this dual admittance of the sign of $\bar{\alpha}$ was termed 'quasi-parity' and was characterized by the quantum number $q = \pm 1$. This in turn also accommodates the second set of bound-state solutions for the PT-symmetric Scarf II potential. It was further noted that the SUSY partner potentials corresponding to equation (46) while turning out to be functions of q, also restore the shape of the same potential. In this way they have extended the concept of shape invariance to PT-symmetric potentials. In the present work, however, the concept of shape invariance is generalized (cf equation (28)) for complex potentials in a very general manner (cf equation (1)) and applied the same to a variety of potentials. The Scarf II potential investigated here as an example thus has the form

$$V(x) = Q_1^2 + \tfrac{1}{4}\left(Q_3^2 - 4Q_1^2 - 4\alpha Q_1\right)\mathrm{sech}^2\alpha x - \left(Q_1 + \tfrac{1}{2}\alpha\right)Q_3\,\mathrm{sech}\,\alpha x \tanh \alpha x \tag{47}$$

where x is given by (1). It is interesting to note that using the generalized definition of shape invariance (cf equation (28)), potential (47), when expressed in terms of real and imaginary parts (cf equation (43)), also exhibits some sort of symmetry under the transformation $Q_1 \to -Q_1$ and $Q_3 \to -Q_3$ in the same way as it manifests in the work of Levai and Znojil [36] with regard to the study of SUSY partner potentials. In other words, such a reflection in the parameter space is a generalization of the concept of 'quasi-parity' which is now extended to complex potentials in general. Further, the shape invariance discussed in the present work also represents a generalization of the concept discussed in the literature for PT-symmetric potentials.

Recently, the study of PT-symmetric non-Hermitian SUSY QM has been carried out by Znojil *et al* [10] by generalizing Witten's quantum-mechanical construction. In this approach the operators $A^{(\pm)}$ and $B^{(\pm)}$ are defined as

$$A^{(\pm)} = \frac{\mathrm{d}}{\mathrm{d}x} + W^{(\pm)}(x) \qquad B^{(\pm)} = -\frac{\mathrm{d}}{\mathrm{d}x} + W^{(\pm)}(x) \tag{48a}$$

which conform to relations furnished by the Reccati equation

$$H^{(\pm)} = B^{(\pm)}A^{(\pm)} = -\partial_x^2 + [W^{(\pm)}(x)]^2 - [W^{(\pm)}(x)]$$

such that the newly defined supercharges are not correlated by any Hermitian conjugation. Using the above result in the Schrödinger equation, they established a correlation between the energy eigenvalues for the *m*th and the *n*th states of the superpartner Hamiltonians as

$$E_m^{(+)} = E_n^{(-)}. \tag{48b}$$

In this formalism, the ordering of the levels is not explicitly spelt out; nor is there a demonstration of any relationship between the energy levels of successive excited states of the superpartner Hamiltonians. It is pertinent to mention here that while in the present work the Witten conditions have been suitably generalized to accommodate all the non-Hermitian Hamiltonian systems (cf equation (18)), the corresponding relation between the eigenvalues of successive eigenstates of the superpartner Hamiltonians has been explicitly demonstrated in equations (25) and (26). Evidently PT-symmetric SUSY QM turns out to be a special case of our present generalized formulation of non-Hermitian SUSY QM.

The SUSY QM for a class of pseudo-Hermitian systems is attempted by Mostafazadeh [29] in which the pseudo-supersymmetric partner Hamiltanians $H^{(+)}$ and $H^{(-)}$ are defined in terms of an operator D as

$$H^{(+)} = \tfrac{1}{2}D^{\#}D \qquad H^{(-)} = \tfrac{1}{2}DD^{\#} \tag{49a}$$

where D turns out to be a linear operator satisfying the relation

$$D^{\#} = \eta_+^{-1}D^{+}\eta_- \tag{49b}$$

with $\eta_\pm$ being the linear, Hermitian and invertible operator, and D^+ being the pseudo-adjoint of D. Further, the intertwining relations are obtained between the superpartner Hamiltonians and the operator D as

$$DH^{(+)} = H^{(-)}D \qquad D^{\#}H^{(-)} = H^{(+)}D^{\#} \tag{50}$$

implying the existence of isospectrality of $H^{(+)}$ and $H^{(-)}$ and hence displaying identical degeneracy structure except for the zero eigenvalue. The above result is also manifest in the present work (cf equations (25) and (26)) although the real and imaginary parts of the operators are related to the superpartner Hamiltonians in a peculiar fashion via equation (16). The intertwining relations obtained in our case are valid for all the non-Hermitian systems. These relations throw some light on the relation between the real and imaginary parts of the operators A and A^+ with the corresponding parts of the supercharges W (cf equations (21) and (22)) which account for the degeneracy structure of real and imaginary parts of the superpartner potentials, leading to a type of correspondence of energy eigenvalues depicted in equations (25) and (26). In yet another interesting paper, Cannata *et al* [28] have provided an explicit framework to study the two-dimensional SUSY QM for pseudo-Hermitian systems with special reference to the particular class of generalized two-dimensional Morse potentials. Contrarily, in our general formulation these two-dimensional attributes result automatically from the complex phase space plane described by equation (1). Within our scheme, the generalized complex Morse potential is shown to be shape invariant with certain reflections in the parameter space (cf equation (37)).

7. Orthonormality of eigenfunctions of non-Hermitian operators

In Schrödinger quantum mechanics (SQM), the wavefunction is a complex function of real variables and it is so designed that it is compatible with all the properties of the Hilbert space and for all the Hermitian operators. The problem of orthonormality of eigenfunctions corresponding to a non-Hermitian operator such as the PT-symmetric one has been discussed recently by several authors [23–25, 40]. The problem of violation of unitarity in this approach has also been resolved by Bender *et al* [26] by introducing a new symmetry, the so-called C-symmetry, in the system. This retrieves the Hermitian character of the Hamiltonian and consequently retains all the good old properties of SQM. Of course, this has been demonstrated only for a particular class of complex potential functions.

In spite of achieving the above features in the PT-symmetric QM, the problem pertaining to the orthonormality of eigenfunctions and that of boundary conditions need to be understood not only in this framework but also for the general case of non-Hermitian operators. As a matter of fact, the Hermitian and non-Hermitian properties of a Hamiltonian are two of its extreme features. In between, while the PT-symmetric version represents a smaller degree of departure from Hermiticity, the pseudo-Hermitian version however represents a larger departure. Clearly, the PT-symmetric Hamiltonians are special cases of pseudo-Hermitian Hamiltonians, which, in turn, are special cases of non-Hermitian ones. The present approach, however, deals with a non-Hermitian case, where the degree of departure from Hermiticity is maximal. In what follows, we briefly outline various attempts made in this direction and discuss a possible way out for these problems in the present approach.

In the framework of Bender and his co-workers [14, 39], the emphasis has been placed on the specific case of PT-symmetric Hamiltonians. In fact, the eigenstates for the PT-symmetric Hamiltonians which are complex, well-behaved in $(-\infty, \infty)$ and asymptotically vanishing on the real line are normalizable. In this case, the real x is replaced with a contour in the complex plane along which the Schrödinger differential equation holds and subsequently the imposed boundary conditions lead to quantization at the end points of the contour via a WKB-type approach. Further, for the regions in the cut complex x-plane (where $\psi(x)$ vanishes asymptotically as $|x| \to \infty$), Bender *et al* [14, 39] have used the concept of wedges bounded by Stokes lines in their treatment inspired by example-based discussion.

In another case, with regard to the discrete spectrum for the PT-symmetric potential of the type

$$V(x) = -(V_1 \operatorname{sech} x + V_2 \tanh x) \operatorname{sech} x \qquad V_1 > 0. \tag{51}$$

Ahmed [40] has proposed a prescription according to which the orthogonality of states $\psi_1(x), \psi_2(x)$ corresponding to the eigenvalues E_1 and E_2 for this potential is defined via the relation

$$\int_{-\infty}^{\infty} \psi_1(x)\psi_2(x)\,\mathrm{d}x = 0 \tag{52}$$

for $E_1 \neq E_2$. Here note the absence of complex conjugation in (52). In view of the involvement of PT operators, equation (52) is recast as

$$\int_{-\infty}^{\infty} \psi_1^{\mathrm{PT}}(x)\psi_2(x)\,\mathrm{d}x = 0 \tag{53}$$

for $E_1^* \neq E_2$, i.e., for the case of PT-symmetric Hamiltonians. On the other hand, in the context of pseudo-Hermitian Hamiltonians one introduces [27] the concept of η-orthogonality, a condition described by [23, 27]

$$(E_i^* - E_j)\int_{-\infty}^{\infty} \psi_i^*(x)\eta\psi_j(x)\,\mathrm{d}x = 0 \tag{54}$$

where η is defined in section 1. Note that condition (53) corresponding to the PT-symmetric case turns out to be a special case of (54). The orthogonality of eigenstates corresponding to the case of relaxed η-pseudo Hermiticity has also been discussed recently by Bagchi and Quesne [41].

Among other approaches to resolve the problem of normalization for at least PT-symmetric Hamiltonians, Bagchi *et al* [24] and Japaridze [25] have exploited the equation of continuity in QM. In this case, Bagchi *et al* [24] make use of a generalized version of the equation of continuity to derive a normalization condition of bound-state eigenfunctions which conform to this new type of (probability) conservation law. On the other hand, Japaridze [25] examines the eigenfunctions of PT-symmetric QM within the framework of an indefinite metric where he uses the concept of Krien space which is decomposed into an orthogonal sum of two Hilbert spaces with positively and negatively defined scalar products. Starting with a relation like equation (53), the inner product of eigenstates ψ_α and ψ_β corresponding to the complex eigenvalues E_α and E_β^* is defined in two different ways. In the first case, noting the fact that the transition probability between the eigenstates with different eigenvalues vanishes in PT-symmetric QM, the inner product is written as

$$(\psi_\alpha|\psi_\beta) = \int_R \psi_\alpha(x)(\theta\psi_\beta(x)) = \int_R \psi_\alpha(x)\psi_\beta^*(-x) \tag{55}$$

where $\psi(x)$ and $\psi^*(-x)$ are solutions of the Schrödinger equation for the potential satisfying

$$\theta V(x) = V^*(-x) = V(x)$$

with the PT-symmetry operator (θ) defined as $\theta(i, \hat{x}, \hat{p})\theta^{-1} = \{-i, -\hat{x}, \hat{p}\}$. In the second case, the inner product, however, is postulated as

$$(\psi_\alpha|\psi_\beta) = \int_R \psi_a(x)(\Omega\psi_\beta(x))\,\mathrm{d}x \tag{56}$$

where Ω is an arbitrary transformation under which the Sturm–Liouville operator H and eigenvalue E transform as $\Omega H\Omega^{-1} = \hat{H}$ and $\Omega E = E$. For the Hermitian case, however, the inner product (56) is required to reduce to a Hermitian case by setting in $V(x) = 0$. With these considerations, finally the discussion of orthonormality of the eigenstates ends up with generalizing the equation of continuity as done in [24]. It may be mentioned that with regard to the boundary conditions on the eigenfunction $\psi(x)$ in the present approach, the results obtained [32, 33] conform to $\lim_{|x|\to\infty}\psi(x) = 0$, at least for the bound states.

One possible way out of the problem of orthogonality of eigenfunctions is by having recourse to the use of polar representation of the complex variable x, namely $x = |x|\exp(\mathrm{i}\theta)$. In the present case, note the difference between the four possible situations furnished by $\psi(x)$, $\psi^*(x)$, $\psi(x^*)$ and $\psi^*(x^*)$. In what follows we shall demonstrate that if one works along the fixed θ-direction, then some simplification with regard to orthogonality of eigenstates in the present treatment can indeed be achieved. In fact, it is found that in this case, the generalized equation of continuity dictates the same results as obtained by Bagchi *et al* [24] and Japaridze [25].

Using $x = |x|\exp(\mathrm{i}\theta)$ or equivalently $\frac{\partial}{\partial x} = \frac{|x|}{x}\frac{\partial}{\partial|x|} - \frac{\mathrm{i}}{x}\frac{\partial}{\partial\theta}$, the analogous Schrödinger equation ($h/2\pi = m = 1$)

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x) \tag{57}$$

takes the form

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\left(\frac{|x|}{x}\frac{\partial}{\partial|x|} - \frac{\mathrm{i}}{x}\frac{\partial}{\partial\theta}\right)\left(\frac{|x|}{x}\frac{\partial}{\partial|x|} - \frac{\mathrm{i}}{x}\frac{\partial}{\partial\theta}\right)\psi(x,t) + V(x)\psi(x,t). \tag{58}$$

In view of the boundary conditions satisfied by the eigenfunctions, $\lim_{|x|\to\infty} \psi(x) = 0$, it is reasonable to suggest that the argument θ has effectively no role to play in determining the orthogonality of the given eigenfunctions. Hence, the orthogonality condition of $\psi(x)$ is invariant with respect to θ and consequently, this condition holds true along any ray in the complex plane. In that case, the terms involving θ-variations in (58) vanish and one writes the analogous Schrödinger equation (the word 'analogous' prefixing the Schrödinger equation is elaborated in our earlier work (see [33])) as

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\frac{|x|}{x}\frac{\partial}{\partial|x|}\left(\frac{|x|}{x}\frac{\partial\psi(x,t)}{\partial|x|}\right) + V(x)\psi(x,t). \tag{59}$$

Now corresponding to equation (59) there are three different ways to write the complex conjugate equations, namely (i) under the change $x \to x^*$ equation (59) leads to

$$\mathrm{i}\frac{\partial\psi(x^*,t)}{\partial t} = -\frac{1}{2}\frac{|x|}{x^*}\frac{\partial}{\partial|x|}\left(\frac{|x|}{x^*}\frac{\partial\psi(x^*,t)}{\partial|x|}\right) + V(x^*)\psi(x^*,t) \tag{60a}$$

(ii) complex conjugation of (59) (in which only the function of x undergoes the process of conjugation along with $\mathrm{i} \to -\mathrm{i}$) gives

$$-\mathrm{i}\frac{\partial\psi^*(x,t)}{\partial t} = -\frac{1}{2}\frac{|x|}{x}\frac{\partial}{\partial|x|}\left(\frac{|x|}{x}\frac{\partial\psi^*(x,t)}{\partial|x|}\right) + V^*(x)\psi^*(x,t) \tag{60b}$$

and (iii) the combination of (i) and (ii) yields

$$-\mathrm{i}\frac{\partial\psi^*(x^*,t)}{\partial t} = -\frac{1}{2}\frac{|x|}{x^*}\frac{\mathrm{d}}{\partial|x|}\left(\frac{|x|}{x^*}\frac{\partial\psi^*(x^*,t)}{\partial|x|}\right) + V^*(x^*)\psi^*(x^*,t). \tag{60c}$$

We now compute the 'probability density analogue' (PDA) for the above three cases. As a matter of fact the above three situations correspondingly impose certain restrictions on the complex potential function, $V(x)$. This enables us to compute the probability density as follows:

Case I. When $V(x) = -V(x^*)$: multiplying (59) by $\psi(x^*,t)$ and (60*a*) by $\psi(x,t)$ and adding the resultant expressions, one immediately obtains

$$\frac{\partial(\psi(x,t)\psi(x^*,t))}{\partial t} = -\frac{1}{2\mathrm{i}}[\psi(x^*,t)D^2\psi(x,t) + \psi(x,t)D^{*2}\psi(x^*,t)] \tag{61}$$

where $D = \frac{|x|}{x}\frac{\partial}{\partial|x|}$ and $D^* = \frac{|x|}{x^*}\frac{\partial}{\partial|x|}$. If we assume that $\psi(x,t)$, $\psi(x^*,t)$ and their derivatives with respect to $|x|$ tend to zero as $|x| \to \infty$ as is the case with bound states, then equation (61) defines the PDA for this case as

$$P_1 = \psi(x,t)\psi(x^*,t). \tag{62}$$

Case II. When $V(x) = V^*(x)$: as in case I multiply equation (59) by $\psi^*(x,t)$ and (60*b*) by $\psi(x,t)$ and subtract the resulting expressions, leading to

$$\begin{aligned}\frac{\partial}{\partial t}(\psi(x,t)\psi^*(x,t)) &= -\frac{1}{2\mathrm{i}}[\psi^*(x,t)D^2\psi(x,t) - \psi(x,t)D^2\psi^*(x,t)]\\ &= D\left[-\frac{1}{2\mathrm{i}}\{\psi^*(x,t)D\psi(x,t) - \psi(x,t)D\psi(x^*,t)\}\right].\end{aligned} \tag{63}$$

In this case, it is possible to write equation (63) in the standard form of the equation of continuity, namely

$$\frac{\partial P_2}{\partial t} + DS = 0 \tag{64}$$

where the PDA and the 'analogous probability current density' respectively are given by

$$P_2 = \psi(x,t)\psi^*(x,t) \tag{65}$$

$$S = -\frac{1}{2\mathrm{i}}[\psi^*(x,t)D\psi(x,t) - \psi(x,t)D\psi(x^*,t)]. \tag{66}$$

Case III. When $V(x) = V^*(x^*)$: multiplying equation (59) by $\psi^*(x^*,t)$ and (60*c*) by $\psi(x,t)$ and subtracting the resulting expressions, one obtains

$$\frac{\partial}{\partial t}(\psi(x,t)\psi^*(x^*,t)) = -\frac{1}{2\mathrm{i}}[\psi^*(x^*,t)D^2\psi(x,t) - \psi(x,t)D^{*2}\psi^*(x^*,t)]. \tag{67}$$

As before, the right-hand side of (60*c*) tends to zero as $|x| \to \infty$, so that the PDA now becomes

$$P_3 = \psi(x,t)\psi^*(x^*,t). \tag{68}$$

It may be mentioned that these probability density analogues P_1, P_2 and P_3 here are pure mathematical constructions. However, it is not difficult to extract some physical meaning out of them particularly after following the prescription of Bagchi *et al* [24]. In this case, the integration will appear along the radial direction $|x|$ and covering the range from zero to infinity, unlike the PT-symmetric case of Bender *et al* where the variable x remains real and integration limits are from $-\infty$ to $+\infty$. Following the work of Bagchi *et al*, one can label the ψ (which appear in bilinear forms) in P_1, P_2, P_3 corresponding to two eigenstates say ψ_α and ψ_β, with corresponding complex eigenvalues E_α and E_β, and discuss the orthogonality of states by rewriting, say for the case I, ψ as

$$P_1 = \psi_\alpha(x,t)\psi_\beta(x^*,t)$$

$$\psi_\alpha(x,t) = u_\alpha(x)\,\mathrm{e}^{-\mathrm{i}E_\alpha t} \qquad \psi_\beta(x^*,t) = u_\beta(x^*)\,\mathrm{e}^{-\mathrm{i}E_\beta t}. \tag{69}$$

The orthogonality of states for this case can be discussed by considering the integrals (cf [24])

$$\frac{\partial}{\partial t}\int_0^\infty \mathrm{d}|x|\psi_\alpha(x,t)\psi_\beta(x^*,t) = 0$$

leading to [24]

$$\int_0^\infty \mathrm{d}|x|u_\alpha(x)u_\beta(x^*) = 0 \tag{70a}$$

for $E_\alpha \neq E_\beta$. Similarly, the orthogonality conditions for the cases II and III respectively turn out to be

$$\int_0^\infty \mathrm{d}|x|u_\alpha(x)u_\beta^*(x) = 0 \tag{70b}$$

$$\int_0^\infty \mathrm{d}|x|u_\alpha(x)u_\beta^*(x^*) = 0. \tag{70c}$$

As for the alternative prescriptions addressing the problem of orthogonality of states in the present approach, particularly in terms of an equivalent two-dimensional real space (i.e., x_1–p_2 space) we refer to [33]. Further studies are in progress.

8. Discussion and conclusions

Motivated by the applications of the general formulation of SUSY QM in the study of PT-symmetric [11–13, 18–22] and pseudo-Hermitian [29] Hamiltonians, we have demonstrated here the viability of the complex phase space approach to analyse these tools of SUSY QM in a rather straightforward manner without actually disturbing the salient aspects of conventional SUSY QM. This has further suggested a deep insight not only into the methodology of SUSY QM but also into the nature of eigenvalue spectra of a variety of complex potentials. In particular, the superpotentials corresponding to the power, singular and exponential potentials are investigated. Not only this, we also account for the complex nature of the underlying parameters appearing in the potentials. Consequently, the results for the known PT-symmetric systems are expected to appear as special cases of the present general formalism.

We have endeavoured to furnish a generalization of the concept of shape invariance for complex potentials involving not only complex parameters but also in the context of complex phase space. This property of shape invariance of complex potentials is exploited and several of its new features automatically manifest in the present approach which otherwise are not so transparent in conventional SUSY QM (cf section 2). In fact, the real and imaginary parts of the superpartners of a given potential $V(x)$ are now separately found to exhibit reflection properties in the complex parameter space. In this way, the four superpartners, namely $V_{\mathrm{r}}^{(+)}$, $V_{\mathrm{r}}^{(-)}$, $V_{\mathrm{i}}^{(+)}$ and $V_{\mathrm{i}}^{(-)}$, can be used to shed light on the connection among the spectra of four adjacent atoms/nuclei in much the same way as conventional SUSY QM does [7, 42] for four or two adjacent atoms/nuclei. Thus the present considerations hold promise for a much better insight into the dynamics of molecules.

Applications of the complex phase approach via SUSY QM are delineated here to study only the ground state of various systems. As for the applications of this formulation to study the excited states, it can be carried out in an analogous and trivial manner. In fact, for excited states the hierarchy of the Hamiltonians H_i and the operators, A_j, A_j^+ $(j = 1, 2, \ldots)$ satisfy [2] the same set of properties as for the ground-state operators.

In section 7, we have attempted to discuss the orthogonality of eigenfunctions of a non-Hermitian operator in general through equations (70*a*)–(70*c*). With regard to the definition of normalization of eigenstates or that of an inner product in the present approach one can, however, follow the underlying prescription of the recent work of Japaridze [25] carried out for the PT-symmetric case. In fact, Japaridze has argued that the condition of orthogonality of eigenfunctions requires them to belong to a space (called Krein space) with an indefinite metric. In other words, the space is decomposable into an orthogonal sum of two Hilbert spaces with positively and negatively defined scalar products. In the same spirit, for the results of section 7 (70), we can define the inner product as

$$(u_\alpha \mid u_\beta) = \int_0^\infty \mathrm{d}|x|u_\alpha(x)(\Omega u_\beta(x))$$

where $\Omega u_\beta(x) = u_\beta(x^*)$, $\Omega u_\beta(x) = u_\beta^*(x)$ and $\Omega u_\beta(x) = u_\beta^*(x^*)$ for the three respective cases discussed in the previous section. Clearly, case II here has some correspondence with the PT-symmetric version of the non-Hermitian operator. Cases I and III, on the other hand, require further analysis in terms of a generalization of the Ω-operator and the rich features of the Krein space [25].

Before closing, it is pertinent to mention that our approach of having invoked the notion of complex phase space within the general framework of SUSY QM has lent tremendous credence to investigate non-Hermitian Hamiltonians and obtained their solutions in a rather straightforward manner. The formulation has the intrinsic merit whereby its applications to a variety of dynamical systems can be studied in a systematic and unambiguous framework whose solutions are otherwise not amenable in the conventional (real) SUSY QM.

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